SYNOPSIS

Title : "SOME INVESTIGATIONS INTO COMMUTATIVE RINGS AND THEIR BROADER IMPLICATIONS FOR RING THEORY"

(a) Introdution

Modern algebra is a important branch of mathematics concerned with the study of algebraic structures such as groups, rings, fields, modules, vector spaces and algebras instead of emphasizing numerical computation. The subject underpins numerous areas of mathematics-including number theory, topology, geometry and representation theory-and has significant applications in fields such as theoretical computer science, cryptography and mathematical physics. We propose to explore the internal structure of rings their associated modules and specialized classes such as group rings and division rings. Special attention is given to commutative rings, which hold central importance in applications throughout algebraic geometry and algebraic number theory.

We propose to further develop ring theory, which provides an abstract framework for analyzing diverse mathematical structures that share fundamental properties. As a foundational discipline, ring theory supports significant applications in number theory, algebraic geometry, topology, cryptography and coding theory.

(b) Work Plan

We propose to develope fundamental concepts of rings, ideals, homomorphisms, modules and modern developments. The motivation for the proposed study and its connections to algebraic structures and applications. We will critically examine existing research in both commutative and non-commutative ring theory and aim to extend results related to group rings, division rings, and module theory. The study also examines applications in algebraic geometry, number theory, and coding theory. Further we investigate Artinian and Noetherian rings along with other specialized algebraic structures.

We theoretically delineate the specific problems and classes of rings to be examined, formulate the corresponding hypotheses and objectives,

and outline the anticipated contributions. We also specify the algebraic structures and methodological tools to be employed, including homological techniques, ideal theory, and aspects of representation theory.

N. Jacobson was widely recognized for his profound contributions to the theory of rings and Lie algebras. His treatise *Algebra* reflects this specialization, providing thorough and insightful coverage of these areas. The foundational material in the work includes essential topics such as group theory, rings, modules, field theory, linear algebra, and Galois theory. We propose to extend established equivalence results among C*-algebras, von Neumann algebras Quantum group and Hopf algebra and further formulate C*-algebraic frameworks for certain locally compact algebras and non-commutative von Neumann algebras. Runde and M. Cowling demonstrates that modern operator algebras are generally non-contractible and that ring-theoretic approaches encounter additional difficulties when such algebras emerge from module-theoretic constructions. We also extend the propoerties of Galois theory brought together previously separate concepts in algebra and introduced powerful abstract techniques that continue to shape modern mathematics. It provided the foundation for major advances in algebraic number theory and algebraic geometry, fundamentally changing how mathematicians study equations and symmetry. This is accomplished through the Fundamental Theorem of Galois Theory, which provides a one-to-one correspondence between the subgroups of the relevant Galois group and the intermediate field extensions associated with the polynomial.

(c) Details of the project with chapterisation

We study addresses the structure of non-commutative rings without imposing finiteness conditions. Many of the results found in Theory of Rings and in the work of Artin, Nesbitt, and Thrall on 'Rings with Minimum Condition' particularly those concerning semisimple rings have been significantly generalized. These broader formulations of earlier concepts have been instrumental in liberating ring-structure theory from dependence on chain conditions. Subsequently, numerous substantial

contributions have been made to the development of ring theory. Ring theory is organized by additional axioms or structural properties. This classification provides a framework for understanding the broad spectrum of algebraic structures that generalize the integers. Non-commutative ring theory, in particular, studies rings in which the order of multiplication is significant. Ring theory also encompasses structures with applications in functional analysis and other fields, including:

- Division Rings: Non-commutative rings with unity in which every nonzero element has a multiplicative inverse.
- C*-Algebras: Rings of operators on complex Hilbert spaces, playing a central role in functional analysis and quantum mechanics.
- Matrix Rings: Non-commutative rings fundamental to linear algebra and representation theory.
- Simple Rings: Non-zero rings whose only ideals are the zero ideal and the ring itself; the structure of simple non-commutative rings is characterized by the Artin-Wedderburn Theorem.
- Boolean Rings: Rings in which every element is idempotent ($x^2 = x$), corresponding to the algebraic framework of logic.

The proposed research project is structured into four chapters, with each chapter emphasizing the extensive applications of the subject across related areas of mathematics. The chapters are outlined as follows:

Chapter - 1 Non-Commutative Ring Theory and Its Structural Properties in Modern Algebra

Ring theory is a branch of modern abstract algebra that focuses on the study of rings algebraic structures that generalize the arithmetic of integers and polynomials. The structural properties of rings enable mathematicians to classify and analyze them in a systematic hierarchy, ranging from broad general classes to highly specialized and well-behaved structures.

Chapter - 2 Structural Characteristics of Prime and Semiprime Rings and Their Impact on Galois Theory and Its Applications

Prime and semiprime rings are fundamental notions in non-commutative

ring theory, extending the idea of integral domains to non-commutative contexts. These concepts play a central role in understanding the structural theory of general rings.

Chapter - 3 The structural features of Artinian and Noetherian rings and their impact on the Jacobson radical

Artinian and Noetherian rings are fundamental notions in commutative algebra that classify rings according to specific conditions imposed on their ideals. These conditions, known as chain conditions, are named after the mathematicians Emil Artin and Emmy Noether.

Chapter - 4 A Study of the Structural Relationship Between Lie Algebras and Hopf Algebras and Its Consequences

Lie algebra theory and Hopf algebra theory are closely linked in modern mathematics, most notably through the construction of the universal enveloping algebra. This construction establishes a functorial correspondence between Lie algebras and a particular class of Hopf algebras. The universal enveloping algebra $U(L)$is an associative algebra constructed to embed the Lie algebra L, and it naturally carries the structure of a Hopf algebra.

(d) Hypothesis

Commutative algebra is the branch of abstract algebra dedicated to the study of commutative rings, their ideals, and modules over such rings. It is a foundational area of mathematics that provides the technical tools for both algebraic geometry and algebraic number theory are Commutative Rings, Ideals and Modules and Noetherian Rings. We also propose to establish that affine Noetherian Pi Hopf algebras are Artin–Schelter Gorenstein and to address a classical affineness question concerning the structure of Noetherian Hopf algebras, as posed by Wu and Zhang. This research highlights the significance of infinite-dimensional Hopf algebras that meet natural finiteness conditions, such as being affine and Noetherian, having finite Gelfand–Kirillov dimension, exhibiting Artin–Schelter regularity, or possessing the Calabi–Yau property.

(e) Motivation

The proposed research project is entirely theoretical in nature. We have access to the necessary reference materials in the Central Library of B.N.M. University, along with additional contemporary and classical books and journals. We may also consult resources available in the Science Section of the National Library in Kolkata and at the Indian Statistical Institute.

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28.11.2025
Signature of the Supervisor

Amit Kumar
Signature of the candidate

21-12-2025
Signature of the Head of the University Department

UNIVERSITY DEPTT. OF MATHEMATICS H.O.D. DHEPURA